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# ON A SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE.

BY T. H. GRONWALL.

1. It is the purpose of this note to investigate the problem of integrating the system

$$(1) \quad \begin{aligned} \frac{\partial^2 \theta}{\partial u_1 \partial u_2} + \frac{\partial \theta}{\partial u_1} - \frac{\partial \theta}{\partial u_2} + 3\theta &= 0, \\ \frac{\partial^2 \theta}{\partial u_2 \partial u_3} + \frac{\partial \theta}{\partial u_2} - \frac{\partial \theta}{\partial u_3} + 3\theta &= 0, \\ \frac{\partial^2 \theta}{\partial u_3 \partial u_1} + \frac{\partial \theta}{\partial u_3} - \frac{\partial \theta}{\partial u_1} + 3\theta &= 0, \end{aligned}$$

which occurs in the preceding paper.\* We begin by showing, by means of the Riemann integration method, that a solution  $\theta = \theta(u_1, u_2, u_3)$  of (1) is uniquely determined by its values on the three coördinate axes, viz.,  $\theta(u_1, 0, 0)$ ,  $\theta(0, u_2, 0)$  and  $\theta(0, 0, u_3)$ .

2. To integrate an equation

$$(2) \quad \frac{\partial^2 z}{\partial x \partial y} + a(x, y) \frac{\partial z}{\partial x} + b(x, y) \frac{\partial z}{\partial y} + c(x, y)z = 0$$

by the Riemann method, we form the adjoint equation

$$(3) \quad \frac{\partial^2 t}{\partial \xi \partial \eta} - \frac{\partial}{\partial \xi} (a(\xi, \eta)t) - \frac{\partial}{\partial \eta} (b(\xi, \eta)t) + c(\xi, \eta)t = 0,$$

and determine a solution

$$t = t(\xi, \eta; x, y)$$

by the initial conditions

$$(4) \quad \begin{aligned} t &= e^{\int_x^\xi b(\xi, y) d\xi} && \text{for } \eta = y, \\ t &= e^{\int_y^\eta a(x, \eta) d\eta} && \text{for } \xi = x; \end{aligned}$$

then the solution of (2) which takes the given values  $z(x, 0)$  on the  $x$ -axis and  $z(0, y)$  on the  $y$ -axis is uniquely determined, and is given by

$$z(x, y) = t(x, 0; x, y)z(x, 0) + t(0, y; x, y)z(0, y) - t(0, 0; x, y)z(0, 0)$$

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\* L. P. Eisenhart, Triply conjugate systems with equal point invariants.

$$(5) \quad \begin{aligned} & + \int_0^x \left[ b(\xi, 0) t(\xi, 0; x, y) - \frac{\partial}{\partial \xi} t(\xi, 0; x, y) \right] z(\xi, 0) d\xi \\ & + \int_0^y \left[ a(0, \eta) t(0, \eta; x, y) - \frac{\partial}{\partial \eta} t(0, \eta; x, y) \right] z(0, \eta) d\eta. \end{aligned}$$

3. We now apply this to the first of equations (1). The adjoint equation is

$$\frac{\partial^2 t}{\partial v_1 \partial v_2} - \frac{\partial t}{\partial v_1} + \frac{\partial t}{\partial v_2} + 3t = 0$$

and the conditions (4) become

$$t = e^{\int_{u_1}^{v_1} -dv_1} = e^{u_1 - v_1} \quad \text{for} \quad v_2 = u_2,$$

$$t = e^{\int_{u_2}^{v_2} dv_2} = e^{v_2 - u_2} \quad \text{for} \quad v_1 = u_1.$$

Writing  $t = e^{u_1 - u_2} \cdot e^{v_2 - v_1} \varphi$ , the adjoint equation becomes

$$\frac{\partial^2 \varphi}{\partial v_1 \partial v_2} + 4\varphi = 0,$$

with the initial conditions  $\varphi = 1$  for  $v_1 = u_1$  and for  $v_2 = u_2$ . With the notation

$$(6) \quad f_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^n}{n!(n+\nu)!} = \frac{1}{(2\sqrt{x})^\nu} J_\nu(4\sqrt{x}),$$

where  $J_\nu$  is the Bessel function of order  $\nu$ , it is readily shown that

$$(7) \quad \varphi(v_1, v_2; u_1, u_2) = f_0((u_1 - v_1)(u_2 - v_2))$$

satisfies both the adjoint equation and the initial conditions. From (6) and (7) it follows that

$$\frac{\partial}{\partial v_1} \varphi(v_1, 0; u_1, u_2) = 4u_2 f_1(u_2(u_1 - v_1)),$$

$$\frac{\partial}{\partial v_2} \varphi(0, v_2; u_1, u_2) = 4u_1 f_1(u_1(u_2 - v_2)),$$

and (5) gives

$$(8) \quad \begin{aligned} \theta(u_1, u_2, u_3) &= e^{-u_2} \theta(u_1, 0, u_3) + e^{u_1} \theta(0, u_2, u_3) - e^{u_1 - u_2} f_0(u_1 u_2) \theta(0, 0, u_3) \\ &\quad - e^{-u_2} \int_0^{u_1} e^{u_1 - v_1} \cdot 4u_2 f_1(u_2(u_1 - v_1)) \theta(v_1, 0, u_3) dv_1 \\ &\quad - e^{u_1} \int_0^{u_2} e^{v_2 - u_2} \cdot 4u_1 f_1(u_1(u_2 - v_2)) \theta(0, v_2, u_3) dv_2. \end{aligned}$$

4. Since the coefficients in (1) are constant, it follows that  $\theta(0, u_2, u_3)$  must satisfy the second, and  $\theta(u_1, 0, u_3)$  the third of these equations. The application of the Riemann method gives, in the same way as before,

$$(9) \quad \begin{aligned} \theta(0, u_2, u_3) &= e^{-u_3}\theta(0, u_2, 0) + e^{u_2}\theta(0, 0, u_3) - e^{u_2-u_3}f_0(u_2u_3)\theta(0, 0, 0) \\ &\quad - e^{-u_3} \int_0^{u_2} e^{u_2-v_2} \cdot 4u_3f_1(u_3(u_2-v_2))\theta(0, v_2, 0)dv_2 \\ &\quad - e^{u_2} \int_0^{u_3} e^{v_3-u_3} \cdot 4u_2f_1(u_2(u_3-v_3))\theta(0, 0, v_3)dv_3 \end{aligned}$$

and

$$(10) \quad \begin{aligned} \theta(u_1, 0, u_3) &= e^{-u_1}\theta(0, 0, u_3) + e^{u_3}\theta(u_1, 0, 0) - e^{u_3-u_1}f_0(u_3u_1)\theta(0, 0, 0) \\ &\quad - e^{-u_1} \int_0^{u_3} e^{u_3-v_3} \cdot 4u_1f_1(u_1(u_3-v_3))\theta(0, 0, v_3)dv_3 \\ &\quad - e^{u_3} \int_0^{u_1} e^{v_1-u_1} \cdot 4u_3f_1(u_3(u_1-v_1))\theta(v_1, 0, 0)dv_1. \end{aligned}$$

Substituting (9) and (10) in (8), we see that  $\theta(u_1, u_2, u_3)$  is uniquely determined when  $\theta(u_1, 0, 0)$ ,  $\theta(0, u_2, 0)$  and  $\theta(0, 0, u_3)$  are given. We may simplify this solution formally by remarking that, since  $\theta(u_1, u_2, u_3)$  is a linear functional of the initial values, it may be written as the sum of four solutions

$$(11) \quad \theta = \theta_0 + \theta_1 + \theta_2 + \theta_3,$$

where  $\theta_0$  is any particular solution such that  $\theta_0(0, 0, 0) = \theta(0, 0, 0)$ , and the initial values of  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are given by the following table

	$\theta_1$	$\theta_2$	$\theta_3$
on $u_1$ -axis $\theta(u_1, 0, 0) - \theta_0(u_1, 0, 0)$		0	0
“ $u_2$ - “	0	$\theta(0, u_2, 0) - \theta_0(0, u_2, 0)$	0
“ $u_3$ - “	0	0	$\theta(0, 0, u_3) - \theta_0(0, 0, u_3)$ .

5. A solution  $\theta_0$  is readily found by writing

$$(12) \quad \theta_0 = \theta(0, 0, 0)e^{\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3}$$

and substituting in (1), we find the following conditions for the constants  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ :

$$\lambda_1\lambda_2 + \lambda_1 - \lambda_2 + 3 = 0,$$

$$\lambda_2\lambda_3 + \lambda_2 - \lambda_3 + 3 = 0,$$

$$\lambda_3\lambda_1 + \lambda_3 - \lambda_1 + 3 = 0.$$

The general solution of these equations is

$$(13) \quad \lambda_1 = \frac{1}{t}, \quad \lambda_2 = \frac{1+3t}{t-1}, \quad \lambda_3 = \frac{1-3t}{t+1},$$

where  $t$  is arbitrary. For instance, making  $t = \pm i\sqrt{3}$  and adding the two resulting expressions (12), we may write

$$\theta_0 = \theta(0, 0, 0) \cos \sqrt{3}(u_1 + u_2 + u_3).$$

6. Writing  $\theta(u_1, 0, 0) - \theta_0(u_1, 0, 0) = h(u_1)$  so that  $h(0) = 0$ , the solution  $\theta_1(u_1, u_2, u_3)$  has the values  $h(u_1), 0, 0$  on the three coördinate axes. Equation (9) gives  $\theta_1(0, u_2, u_3) = 0$ , (10) becomes

$$\theta_1(u_1, 0, u_3) = e^{u_3}h(u_1) - e^{u_3} \int_0^{u_1} e^{v_1-u_1} \cdot 4u_3 f_1(u_3(u_1 - v_1))h(v_1)dv_1,$$

and substituting in (8), we find

$$\begin{aligned} \theta_1(u_1, u_2, u_3) = & e^{u_3-u_2} \left\{ h(u_1) - 4u_3 \int_0^{u_1} e^{v_1-u_1} f_1(u_3(u_1 - v_1))h(v_1)dv_1 \right. \\ & - 4u_2 \int_0^{u_1} e^{u_1-v_1} f_1(u_2(u_1 - v_1))h(v_1)dv_1 \\ & \left. + 16u_2u_3 \int_0^{u_1} e^{u_1-w_1} f_1(u_2(u_1 - w_1))dw_1 \int_0^{w_1} e^{v_1-w_1} f_1(u_3(w_1 - v_1))h(v_1)dv_1 \right\}. \end{aligned}$$

Changing the order of integration in the repeated integral, this becomes

$$(14) \quad \theta_1(u_1, u_2, u_3) = e^{u_3-u_2} \left[ h(u_1) + \int_0^{u_1} \Phi(u_1 - v_1, u_2, u_3)h(v_1)dv_1 \right]$$

where

$$\begin{aligned} \Phi(u_1 - v_1, u_2, u_3) = & 16u_2u_3 \int_{v_1}^{u_1} e^{v_1-w_1} f_1(u_3(w_1 - v_1))f_1(u_2(u_1 - w_1))dw_1 \\ & - 4u_2e^{u_1-v_1} f_1(u_2(u_1 - v_1)) - 4u_3e^{v_1-u_1} f_1(u_3(u_1 - v_1)) \end{aligned}$$

or, making  $w_1 = v_1 + (u_1 - v_1)x$ ,

$$\begin{aligned} (15) \quad \Phi(u_1 - v_1, u_2, u_3) = & 16u_2u_3(u_1 - v_1) \\ & \times \int_0^1 e^{-(u_1-v_1)x} f_1(u_3(u_1 - v_1)x) f_1(u_2(u_1 - v_1)(1-x))dx \\ & - 4u_2e^{u_1-v_1} f_1(u_2(u_1 - v_1)) - 4u_3e^{v_1-u_1} f_1(u_3(u_1 - v_1)). \end{aligned}$$

The corresponding expressions for  $\theta_2$  and  $\theta_3$  are found by cyclic permutation of the variables.

7. In some cases,  $\theta_1(u_1, u_2, u_3)$  may be expressed as a contour integral. From (12) and (13) it follows that a solution of (1) is given by

$$(16) \quad \theta_1(u_1, u_2, u_3) = \frac{1}{2\pi i} \int_C e^{\frac{u_1}{t} + \frac{1+3t}{t-1}u_2 + \frac{1-3t}{t+1}u_3} f(t)dt,$$

where the contour  $C$  encloses  $t = 0$ , but neither  $t = 1$  nor  $t = -1$ , and  $f(t)$

is holomorphic inside  $C$ . For  $u_1 = 0$ , the integrand is holomorphic inside  $C$ , so that  $\theta_1(0, u_2, u_3) = 0$  and in particular  $\theta_1(0, u_2, 0) = \theta_1(0, 0, u_3) = 0$ . For  $u_2 = u_3 = 0$ , we find

$$h(u_1) = \theta_1(u_1, 0, 0) = \frac{1}{2\pi i} \int_C e^{u_1 t} f(t) dt.$$

Let  $f(t) = \sum_0^\infty c_\nu t^\nu$  converge for  $|t| < r$  and deform  $C$  into the circle  $|t| = \rho < r$ ; then

$$h(u_1) = \frac{1}{2\pi i} \int_{|t|=\rho} \sum_{\mu, \nu=0}^\infty \frac{c_\nu u_1^\mu}{\mu!} t^{\nu-\mu} dt = \sum_{\nu=0}^\infty \frac{c_\nu u_1^{\nu+1}}{(\nu+1)!}.$$

When

$$h(u_1) = \sum_{\nu=1}^\infty a_\nu u_1^\nu$$

is given, it follows that  $c_\nu = (\nu+1)!a_{\nu+1}$  and

$$(17) \quad f(t) = \sum_{\nu=0}^\infty (\nu+1)!a_{\nu+1}t^\nu,$$

provided that this series has a radius of convergence greater than zero. For instance, when  $h(u_1) = e^{u_1} - 1$ , then  $f(t) = 1/(1-t)$ .